

A NOTE ON FAST TIMES OF BROWNIAN MOTION WITH VARIABLE DRIFT

JULIA RUSCHER

ABSTRACT. A famous result of Orey and Taylor gives the Hausdorff dimension of the set of fast times, that is the set of points where linear Brownian motion moves faster than according to the law of iterated logarithm. In this paper we examine what happens to the set of fast times if a variable drift is added to linear Brownian motion. In particular, we will show that the Hausdorff dimension of the set of fast times cannot be decreased by adding a function to Brownian motion.

1. INTRODUCTION

Let $B(t)$ be standard one-dimensional Brownian motion with $B(0) = 0$. In 1974 Orey and Taylor [OT74] studied the so-called fast points of Brownian motion. That is, for a given $a > 0$ a time $t \in [0, 1]$ with

$$\limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} \geq a$$

is called an *a-fast time* of linear Brownian motion. By the law of iterated logarithm follows that the set of *a-fast times* has Lebesgue measure zero. Therefore, to quantify how often these *a-fast times* occur we use Hausdorff dimension. Orey and Taylor [OT74] showed for every $a \in [0, 1]$,

$$\dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = 1 - a^2, \quad (1)$$

almost surely.

Khoshnevisan and Shi ([KS]) extended Orey's and Taylor's results [OT74] in several different ways. One of which is the intersection of the set of fast points with the zero set of Brownian motion.

Theorem 1.1 ([KS]). *Let $\mathcal{Z}(B) := \{t \in (0, 1] \mid B(t) = 0\}$. For every $a \in (0, 1]$*

$$\dim \left\{ t \in \mathcal{Z}(B) \mid \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = \max \left\{ \frac{1}{2} - a^2, 0 \right\}$$

almost surely.

Key words and phrases. Brownian motion, fast times, Hausdorff dimension.

In the present note we will first give some general remarks on fast times of Brownian motion with variable drift, see section 2. We can extend the result of Orey and Taylor by adding a continuous function f to Brownian motion and giving a general lower bound on the Hausdorff dimension of the set of a -fast times. In particular, this theorem implies that by adding a function to Brownian motion the Hausdorff dimension of the set of a -fast times cannot be decreased.

Theorem 1.2. *Suppose $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is an arbitrary function and $X(t) := B(t) - f(t)$. For every $a \in (0, 1]$*

$$\dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} \geq 1 - a^2,$$

almost surely.

An example of a function f where the dimension of a -fast times is strictly greater than $1 - a^2$ is given in the next section (see Proposition 2.2 and the subsequent example). The following result is an upper bound analogue of Theorem 1.1 for $1/2$ -Hölder continuous functions added to one-dimensional Brownian motion. Note that the Hausdorff dimension of the zero set of the latter is $1/2$, see Corollary 1.7 of [ABPR].

Theorem 1.3. *Suppose $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a $1/2$ -Hölder continuous function, $X(t) := B(t) - f(t)$ and let $\mathcal{Z}(X) := \{t \in (0, 1] \mid X(t) = 0\}$. For every $a \in (0, 1]$*

$$\dim \left\{ t \in \mathcal{Z}(X) \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} \leq \max \left\{ \frac{1}{2} - a^2, 0 \right\}$$

almost surely.

We will prove the upper using the method of [KS] in section 3. A general lower bound for continuous functions can be given as well.

Theorem 1.4. *Suppose $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function, $X(t) := B(t) - f(t)$ and let $\mathcal{Z}(X) := \{t \in (0, 1] \mid X(t) = 0\}$. For every $a \in (0, 1]$*

$$\dim \left\{ t \in \mathcal{Z}(X) \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} \geq \max \left\{ \frac{1}{2} - a^2, 0 \right\}$$

with positive probability.

2. SOME FIRST REMARKS ON FAST TIMES OF BROWNIAN MOTION WITH VARIABLE DRIFT

By the Cameron-Martin theorem (see Theorem 1.38 in [MP] or Theorem 2.2 in Chapter 8 in [RY]) we see that the Theorem of Orey and Taylor, see (1), holds as well if we replace Brownian motion by a function f added to Brownian motion where f is in the Cameron-Martin space $\mathbf{D}(I)$ (integrals of functions in $\mathbf{L}^2(I)$). We will show that the same holds for any function f which is locally $1/2$ -Hölder

continuous. Because all functions in $\mathbf{D}(I)$ are $1/2$ -Hölder continuous, this is a stronger statement than the one implied by the Cameron-Martin theorem.

Corollary 2.1. *Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a locally $1/2$ -Hölder continuous function and let $X(t) := B(t) - f(t)$. Then, for every $a \in [0, 1]$*

$$\dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = 1 - a^2$$

almost surely.

Proof. By the definition of $1/2$ -Hölder continuity and the triangle inequality we get that for every $t \geq 0$,

$$\begin{aligned} \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} &= \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)| - |f(t+h) - f(t)|}{\sqrt{2h \log(1/h)}} \\ &\leq \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \\ &\leq \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)| + |f(t+h) - f(t)|}{\sqrt{2h \log(1/h)}} \\ &= \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}}. \end{aligned}$$

□

Note that the statement of corollary 2.1 also holds if f is not locally $1/2$ -Hölder continuous on a countable subset of \mathbb{R}^+ .

A natural question to ask is if we can perturb linear Brownian motion by a function such that the Hausdorff dimension of the set of a -fast times differs from the result (1). The following proposition gives a positive answer. Also, this is an example for a function such that a strict inequality holds in Theorem 1.2 (for some a). More examples are given below.

Proposition 2.2. *Let $f_\gamma: [0, 1] \rightarrow [0, 1]$ be a middle $(1 - 2\gamma)$ -Cantor function with $\gamma < 1/4$ and let $X_\gamma(t) := B(t) - f_\gamma(t)$. Then, for every $a \in [0, 1]$*

$$\dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|X_\gamma(t+h) - X_\gamma(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = \max \left\{ 1 - a^2, -\frac{\log 2}{\log \gamma} \right\},$$

almost surely.

Note here that $-\frac{\log 2}{\log \gamma}$ is also both the Hausdorff dimension of the Cantor set and the Hölder exponent of the Cantor function.

Proof. For $n > 0$ we call the n -th approximation of the Cantor set $C_{\gamma,n}$, $\mathfrak{C}_{\gamma,n}$ the set of all connected components of $C_{\gamma,n}$, and $f_{\gamma,n}$ the corresponding n -th approximation of the Cantor function, see e.g. [ABPR], section 3 for precise definition. Take an arbitrary $\gamma < \gamma_1 < 1/4$. There is an $n_0 > 0$ such that $\sum_{n \geq n_0} (2\sqrt{\gamma_1})^n \leq$

1/2. For $n \geq n_0$ consider the interval $J_{k,n} = [k2^{-n} - \gamma_1^{n/2}/2, k2^{-n} + \gamma_1^{n/2}/2]$ and define the set $M_{n_0} = \bigcup_{n \geq n_0} \bigcup_{0 \leq k \leq 2^n} J_{k,n}$. Take $t \in C_\gamma \setminus f_\gamma^{-1}(M_{n_0})$ and any $s \neq t$ in the same connected component of the interior of C_{γ,n_0} . The largest integer ℓ such that both s and t are contained in the same interval of $\mathfrak{C}_{\gamma,\ell}$ satisfies $\ell \geq n_0$. Moreover, $|f_\gamma(s) - f_\gamma(t)| \geq \gamma_1^{(\ell+1)/2}$ and $|s - t| \leq \gamma^\ell$. We see that t satisfies

$$\limsup_{h \downarrow 0} \frac{|f_\gamma(t+h) - f_\gamma(t)|}{h^\beta} > 0.$$

with $\beta = \frac{\log \gamma_1}{2 \log \gamma} < 1/2$. Hence, t is an a -fast time of the process X_γ .

Because $\sum_{n \geq n_0} (2\sqrt{\gamma_1})^n \leq 1/2$ note that for every n holds

$$|C_{\gamma,n} \setminus f_\gamma^{-1}(M_{n_0})| \geq 1/2 |C_{\gamma,n}|.$$

Therefore the Hausdorff dimension of the fast times of the process X_γ on the set $C_\gamma \setminus f_\gamma^{-1}(M_{n_0})$ equals the Hausdorff dimension of the Cantor set (that is $-\frac{\log 2}{\log \gamma}$).

The Hausdorff dimension of fast times on the set $[0, 1] \setminus C_\gamma$, that is the union of open intervals where the function f_γ is constant, is $1 - a^2$. Note, that the set $f_\gamma^{-1}(M_{n_0}) \cap C_\gamma$ has at most the Hausdorff dimension $-\frac{\log 2}{\log \gamma}$. Then, by the definition of Hausdorff dimension we see that for two sets A and B it holds $\dim(A \cup B) = \sup\{\dim A, \dim B\}$. The claim follows. \square

Note that there are functions such that for all $a > 0$ the Hausdorff dimension of the set of a -fast times of these functions added to Brownian motion is 1 almost surely. For instance, Loud in [Loud] constructed functions which satisfy a certain local reverse Hölder property at each point (see also the construction in [MP53]). These functions are defined as $g(t) = \sum_{k=1}^{\infty} g_k(t)$ where $g_k(t) = 2^{-2A\alpha k} g_0(2^{2Ak}t)$, for $0 < \alpha < 1$, a positive integer A such that $2A(1 - \alpha) > 1$, and a continuous function g_0 which has value 0 at even integers, value 1 at odd integers and is linear at all other points. It holds that there is a positive constant c such that $|g(t+h) - g(t)| > ch^\alpha$ for infinitely many arbitrarily small $h > 0$ (see Theorem of [Loud]). Therefore, if we choose $\alpha < 1/2$, then for every $a \geq 0$,

$$\dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|(B - g)(t+h) - (B - g)(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\} = 1,$$

almost surely.

For fractional Brownian motion Khoshnevisan and Shi ([KS]) proved the following analogue result to (1).

Theorem 2.3 ([KS]). *Let $B^{(H)}: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a fractional Brownian motion with Hurst index $H \in]0, 1[$ and $B^{(H)}(0) = 0$. For every $a \in (0, 1)$*

$$\dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|B^{(H)}(t+h) - B^{(H)}(t)|}{\sqrt{2 \cdot h^H \sqrt{\log(1/h)}}} \geq a \right\} = 1 - a^2$$

almost surely.

As before we can extend this result.

Corollary 2.4. *Let $B^{(H)}: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a fractional Brownian motion with Hurst index $H \in]0, 1[$ and $B^{(H)}(0) = 0$.*

(i) Let $f^{(H)}: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a locally H -Hölder continuous function and let $X^{(H)}(t) := B^{(H)}(t) - f^{(H)}(t)$. Then, for every $a \in [0, 1]$

$$\dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|X^{(H)}(t+h) - X^{(H)}(t)|}{\sqrt{2} \cdot h^H \sqrt{\log(1/h)}} \geq a \right\} = 1 - a^2$$

almost surely.

(ii) Let $f_\alpha: [0, 1] \rightarrow [0, 1]$ be a middle α -Cantor function with $\alpha \in (0, 1)$ and let $X_\alpha^{(H)}(t) := B^{(H)}(t) - f_\alpha(t)$. Then, for every $a \in [0, 1]$, and every $\alpha > 1 - 2^{1-\frac{1}{H}}$

$$\begin{aligned} \dim \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|X_\alpha^{(H)}(t+h) - X_\alpha^{(H)}(t)|}{\sqrt{2} \cdot h^H \sqrt{\log(1/h)}} \geq a \right\} \\ = \max \left\{ 1 - a^2, \frac{\log 2}{\log 2 - \log(1 - \alpha)} \right\} \end{aligned}$$

almost surely.

Proof. Analogously to the proofs of Corollary 2.1 and Proposition 2.2. \square

As we have already mentioned, Khoshnevisan and Shi ([KS]) also looked at the intersection set of a -fast times and the zero set of Brownian motion (see 1.1). Unfortunately, it is not known whether an analogue statement holds for fractional Brownian motion. (The proof cannot be adapted for fractional Brownian motion with $H \neq 1/2$ since the increments of the process are not independent.)

3. THEOREM 1.3: UPPER BOUND

First we denote the set of a -fast times for every $a \in (0, 1]$ by $F(a)$, that is

$$F(a) := \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\}.$$

By the proof of corollary 2.1 we see that,

$$F(a) = \left\{ t \in [0, 1] \mid \limsup_{h \downarrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\}.$$

Further we define for every $a \in (0, 1]$ and $h > 0$,

$$\mathfrak{F}(a, h) := \left\{ t \in [0, 1] \mid \sup_{t \leq s \leq t+h} |B(s) - B(t)| \geq a \sqrt{2h \log(1/h)} \right\}.$$

Then, for all $0 < b < a$, we have that $F(a) \subset \bigcap_{h>0} \bigcup_{0<\delta<h} \mathfrak{F}(b, \delta)$. Now let $I_{k,j}^\eta := [k\beta^{-\eta j}, (k+1)\beta^{-\eta j}]$ for any $\beta, \eta > 1$, all $j \geq 1$, and every integer $0 \leq k < \beta^{\eta j}$. For all $t \in \mathfrak{F}(b, \delta)$ it holds for $\delta < h < 1$ with $\beta^{-j} \leq \delta \leq \beta^{1-j}$ that

$$\begin{aligned} \sup_{t \leq s \leq t + \beta^{1-j}} |B(s) - B(t)| &\geq b\beta^{-j/2} \sqrt{2 \log(\beta^{j-1})} \\ &= b\beta^{-1/2} \beta^{-(j-1)/2} \sqrt{2 \log(\beta^{j-1})}. \end{aligned} \quad (2)$$

It follows $t \in \mathfrak{F}(b\beta^{-1/2}, \beta^{1-j})$. We fix $\beta, \eta > 1$, $\theta \in]0, 1[$, then we get for any integer $i \geq 1$,

$$F(a) \subset \bigcup_{j \geq i} \bigcup_{k \geq 1} I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}).$$

f is a $1/2$ -Hölder continuous function, that is $|f(t) - f(s)| \leq c_0 |t - s|^{1/2}$ for some $c_0 > 0$ and all $s, t \in [0, 1]$. Now we will bound the probability of the event $|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}$ from above with $c_1 := \max\{2c_0, 2\sqrt{2}\}$. By the scaling property of Brownian motion we get,

$$\begin{aligned} \mathbb{P}\{|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\} \\ &= \mathbb{P}\{B(k\beta^{-\eta j}) \in [-c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)} + f(k\beta^{-\eta j}), \\ &\quad c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)} + f(k\beta^{-\eta j})]\} \\ &= \mathbb{P}\{B(1) \in [-c_1 k^{-1/2} \sqrt{\eta j \log(\beta)} + f(k\beta^{-\eta j}) k^{-1/2} \beta^{\eta j/2}, \\ &\quad c_1 k^{-1/2} \sqrt{\eta j \log(\beta)} + f(k\beta^{-\eta j}) k^{-1/2} \beta^{\eta j/2}]\}. \end{aligned}$$

We obtain $0 \leq |f(k\beta^{-\eta j})| k^{-1/2} \beta^{\eta j/2} \leq c_0$. By symmetry and the unimodality property of the normal distribution, we get that

$$\begin{aligned} \mathbb{P}\{|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\} \\ \leq \mathbb{P}\{B(1) \in [-c_1 k^{-1/2} \sqrt{\eta j \log(\beta)}, c_1 k^{-1/2} \sqrt{\eta j \log(\beta)}]\} \\ \leq 2c_1 k^{-1/2} \sqrt{\eta j \log(\beta)}. \end{aligned} \quad (3)$$

Also, note that by Levy's modulus of continuity there exists a finite random variable K , depending on η and β , such that for all $j \geq K$ almost surely $\mathbf{1}_{\{|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}\}} \geq \mathbf{1}_{\{I_{k,j}^\eta \cap \mathcal{Z}(X) \neq \emptyset\}}$. Therefore,

$$F(a) \cap \mathcal{Z}(X) \subset \bigcup_{j \geq i} \bigcup_{k: \substack{|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \\ \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}}} I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}).$$

The next step is to show that this is a good covering. With (3) we get, for any $\gamma > 0$,

$$\begin{aligned}
& \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I_{k,j}^\eta|^\gamma \mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset, I_{k,j}^\eta \cap \mathcal{Z}(X) \neq \emptyset) \\
& \leq \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I_{k,j}^\eta|^\gamma \mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset, \\
& \quad |B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}) \\
& \leq \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I_{k,j}^\eta|^\gamma \mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset) \\
& \quad \cdot \mathbb{P}(|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}),
\end{aligned}$$

where we used the independence of increments of Brownian motion in the last step.

In order to bound $\mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset)$ from above we will need the following Lemma.

Lemma 3.1 (see Lemma 3.1. of [KS]). *For all $b > 0$, $0 < \varepsilon < 1$, $\eta > 1$, and all $\beta > 1$, there is a $2 \leq J < \infty$ depending on ε, η, b and β such that for all $j \geq J$ and all $k \geq 0$,*

$$\mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(b, \beta^{-j}) \neq \emptyset) \leq \beta^{-b^2(1-\varepsilon)j}.$$

Hence, we obtain that for all $\mu \in]0, 1[$ there is a $\infty > J \geq 2$ depending on μ, η, a, β and θ such that for $j \geq J$ and all $k > 0$, $\mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset)$ is bounded from above by $\beta^{-\theta^2 a^2 \beta^{-1}(1-\mu)(j-1)}$. Note that $\beta^{-\theta^2 a^2 \beta^{-1}(1-\mu)(j-1)} \leq \beta^{-\theta^2 a^2 \beta^{-2}(1-\mu)j}$ for large enough j . Thus, for large enough i ,

$$\begin{aligned}
& \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I_{k,j}^\eta|^\gamma \mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset) \\
& \quad \cdot \mathbb{P}(|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}) \\
& \leq \sum_{j \geq i} \beta^{-\eta j} \beta^{-\theta^2 a^2 \beta^{-2}(1-\mu)j} \left(1 + \sum_{k=1}^{\beta^{\eta j}-1} 2c_1 k^{-1/2} \sqrt{\eta j \log(\beta)} \right) \\
& \leq \sum_{j \geq i} \beta^{-\eta j} \beta^{-\theta^2 a^2 \beta^{-2}(1-\mu)j} (\beta^{\eta j/2} \cdot 2c_1 \sqrt{\eta j \log(\beta)} + 1).
\end{aligned}$$

That means, if $\eta\gamma - \eta/2 + \theta^2 a^2 \beta^{-2}(1 - \mu) > 0$, then almost surely

$$\lim_{i \rightarrow \infty} \sum_{j \geq i} \sum_{0 \leq k < \beta^{\eta j}} |I_{k,j}^\eta|^\gamma \mathbb{P}(I_{k,j}^\eta \cap \mathfrak{F}(\theta a \beta^{-1/2}, \beta^{1-j}) \neq \emptyset) \cdot \mathbb{P}(|B(k\beta^{-\eta j}) - f(k\beta^{-\eta j})| \leq c_1 \sqrt{\eta j \beta^{-\eta j} \log(\beta)}) = 0.$$

By letting $\mu \downarrow 0$, $\theta \uparrow 1$, $\beta \downarrow 1$ and $\eta \downarrow 1$ the claim follows. ■

4. PROOF OF THEOREM 1.4

In order to prove Theorem 1.4 we will give a proof of the following theorem which is an analogue of Theorem 8.1. of [KS]. The statement of Theorem 4.1 might be of independent interest.

Theorem 4.1. *Let $E \subset [0, 1]$ be a compact set. If $\dim(E) > a^2 + 1/2$, then the set*

$$\left\{ t \in \mathcal{Z}(X) \cap E \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\}$$

is non-empty with positive probability.

Now the lower bound of Theorem 1.4 follows using the following stochastic codimension argument. For a random set $M \subset \mathbb{R}_+$ the *upper stochastic codimension* $\overline{\text{codim}}(M)$ is defined by the smallest value γ such that for all Borel measurable sets G with $\dim(G) > \gamma$ holds that $\mathbb{P}(G \cap M \neq \emptyset) > 0$. Then $\overline{\text{codim}}(M) + \dim(M) \geq 1$ with positive probability, see [Kho02], p. 436 and also [Kho03], p. 238.

In order to prove Theorem 4.1 we need some technical lemmas. First we give some definitions. For $\eta > 0$ and an atomless probability measure μ , call

$$A_\eta(\mu) := \sup_{0 < h \leq 1/2} \sup_{t \in [h, 1-h]} \frac{\mu[t-h, t+h]}{h^\eta}.$$

Further, define for $h > 0$

$$S_h(\mu) := \sup_{0 \leq s \leq h} \int_s^h \frac{1}{\sqrt{t-s}} d\mu(t), \text{ and}$$

$$\tilde{S}_h(\mu) := \sup_{0 \leq s \leq 1} \int_s^{(s+h \wedge 1)} \frac{1}{\sqrt{t-s}} d\mu(t).$$

The first lemma is a version of the famous Frostman's lemma.

Lemma 4.2 (Frostman, cf. [Kah], p. 130). *Let $\eta > 0$, and let $E \subset [0, 1]$ be Borel measurable set satisfying $\eta < \dim(E)$, then there is an atomless probability measure μ on E for which $A_\eta(\mu) < \infty$.*

Lemma 4.3 ([KS], Lemma 8.2). *Let μ be an atomless probability measure on a compact set $E \subset [0, 1]$, and for $h > 0$ and $\eta > 1/2$,*

$$S_h(\mu) \leq \frac{2 \exp(\eta)}{2\eta - 1} A_\eta(\mu) h^{\eta-1/2},$$

$$\tilde{S}_h(\mu) \leq \frac{2 \exp(\eta)}{2\eta - 1} A_\eta(\mu) h^{\eta-1/2}.$$

Lemma 4.4 ([KS], Theorem 2.5). *Let (E_n) be a countable collection of open random sets. If $\sup_{n \geq 1} \overline{\text{codim}}(E_n) < 1$, then*

$$\overline{\text{codim}}\left(\bigcap_{n=1}^{\infty} E_n\right) = \sup_{n \geq 1} \overline{\text{codim}}(E_n).$$

Proof of Theorem 4.1. First, for $h > 0$ we define the two sets

$$\mathcal{S}^+(h) := \{t \in [0, 1] : f(t+h) - f(t) \geq 0\},$$

and

$$\mathcal{S}^-(h) := \{t \in [0, 1] : f(t+h) - f(t) \leq 0\},$$

Now for an atomless probability measure μ on E let

$$\mathcal{S}^\circ(h) := \begin{cases} \mathcal{S}^-(h), & \text{if } \int_0^1 \mathbf{1}_{\mathcal{S}^-(h)}(s) d\mu(s) \geq \int_0^1 \mathbf{1}_{\mathcal{S}^+(h)}(s) d\mu(s), \\ \mathcal{S}^+(h), & \text{if } \int_0^1 \mathbf{1}_{\mathcal{S}^-(h)}(s) d\mu(s) < \int_0^1 \mathbf{1}_{\mathcal{S}^+(h)}(s) d\mu(s). \end{cases}$$

Since μ is a probability measure on the set $E \subset [0, 1]$ it follows $1 \geq \int_0^1 \mathbf{1}_{\mathcal{S}^\circ(h)} d\mu(s) \geq 1/2$.

Define

$$J_\mu(h, a) := \int_0^1 \mathbf{1}_{\{B(s) \in (f(s)-h, f(s)+h)\}} \cdot \mathbf{1}_{\{B(s+h)-B(s) > a\sqrt{2h \log(1/h)} \text{ if } \mathcal{S}^\circ(h) = \mathcal{S}^-(h), B(s+h)-B(s) < -a\sqrt{2h \log(1/h)} \text{ if } \mathcal{S}^\circ(h) = \mathcal{S}^+(h)\}} d\mu(s).$$

In the following we will denote the event $B(s+h) - B(s) > a\sqrt{2h \log(1/h)}$ if $\mathcal{S}^\circ(h) = \mathcal{S}^-(h)$ and $B(s+h) - B(s) < -a\sqrt{2h \log(1/h)}$ if $\mathcal{S}^\circ(h) = \mathcal{S}^+(h)$ by $\mathcal{K}_a(s, h)$.

For $h > 0$ and $s \in [0, 1]$, there are constants $C_1 > 0$ and $C_2 > 0$ (only depending on $\max_{x \in [0, 1]} |f(x)|$) with

$$C_1 s^{-1/2} h \leq \mathbb{P}(B(s) \in (f(s) - h, f(s) + h)) \leq C_2 s^{-1/2} h. \quad (4)$$

Note that, by the independence of increments of Brownian motion,

$$\begin{aligned} \mathbb{E}(J_\mu(h, a)) &= \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{2\log(1/h)}}^\infty \exp\left(-\frac{u^2}{2}\right) du \int_0^1 \mathbf{1}_{S^\circ(h)}(t) d\mu(t) \\ &\quad \cdot \int_0^1 \mathbb{P}(B(s) \in (f(s) - h, f(s) + h)) d\mu(s) \end{aligned}$$

Applying 4 we get

$$\mathbb{E}(J_\mu(h, a)) \geq \frac{C_1}{2\sqrt{2\pi}} h \int_{a\sqrt{2\log(1/h)}}^\infty \exp\left(-\frac{u^2}{2}\right) du \int_{h^2}^1 s^{-1/2} \mu(ds).$$

We fix an $h' > 0$, then there is a constant $c_1 > 0$ (depending on $\max_{x \in [0,1]} |f(x)|$) for all $0 < h \leq h'$ such that

$$\mathbb{E}(J_\mu(h, a)) \geq c_1 h \int_{a\sqrt{2\log(1/h)}}^\infty \exp\left(-\frac{u^2}{2}\right) du. \quad (5)$$

Later we will apply the second moment method to $J_\mu(h, a)$. Therefore, we need to bound the second moment of $J_\mu(h, a)$ from above.

$$\begin{aligned} \mathbb{E}(J_\mu^2(h, a)) &= 2\mathbb{E}\left[\int_0^1 \mathbf{1}_{\{B(t) \in (f(t)-h, f(t)+h)\}} \mathbf{1}_{\mathcal{K}_a(t, h)} \right. \\ &\quad \cdot \left. \int_0^t \mathbf{1}_{\{B(s) \in (f(s)-h, f(s)+h)\}} \mathbf{1}_{\mathcal{K}_a(s, h)} d\mu(s) d\mu(t)\right] \\ &= \int_0^1 \mathbf{1}_{S^\circ(h)}(r) d\mu(r) \cdot \sqrt{\frac{2}{\pi}} \int_{a\sqrt{2\log(1/h)}}^\infty \exp\left(-\frac{u^2}{2}\right) du \\ &\quad \cdot \mathbb{E}\left[\int_0^1 \mathbf{1}_{\{B(t) \in (f(t)-h, f(t)+h)\}} \int_0^t \mathbf{1}_{\{B(s) \in (f(s)-h, f(s)+h)\}} \mathbf{1}_{\mathcal{K}_a(s, h)} d\mu(s) d\mu(t)\right] \\ &\leq \sqrt{\frac{2}{\pi}} \int_{a\sqrt{2\log(1/h)}}^\infty \exp\left(-\frac{u^2}{2}\right) du \cdot (T_1 + T_2), \end{aligned} \quad (6)$$

where

$$\begin{aligned} T_1 &= \mathbb{E}\left[\int_h^1 \mathbf{1}_{\{B(t) \in (f(t)-h, f(t)+h)\}} \cdot \int_0^{(t-h)^+} \mathbf{1}_{\{B(s) \in (f(s)-h, f(s)+h)\}} \right. \\ &\quad \cdot \left. \mathbf{1}_{\mathcal{K}_a(s, h)} d\mu(s) d\mu(t)\right], \\ T_2 &= \mathbb{E}\left[\int_0^1 \mathbf{1}_{\{B(t) \in (f(t)-h, f(t)+h)\}} \int_{(t-h)^+}^t \mathbf{1}_{\{B(s) \in (f(s)-h, f(s)+h)\}} d\mu(s) d\mu(t)\right]. \end{aligned}$$

First, we will estimate T_1 , note that

$$T_1 = \int_h^1 \int_0^{(t-h)^+} \mathbb{P}\left(B(t) \in (f(t) - h, f(t) + h), B(s) \in (f(s) - h, f(s) + h), \right. \\ \left. \mathcal{K}_a(s, h)\right) d\mu(s) d\mu(t).$$

Take a $t \in [h, 1]$ and an $s \in [0, t - h]$. Then we have $s \leq s + h \leq t$ and,

$$\begin{aligned} \mathbb{P}(B(t) \in (f(t) - h, f(t) + h) | B(r) \text{ with } r \leq s + h) \\ = \mathbb{P}(B(t) - B(s + h) + B(s + h) \in (f(t) - h, f(t) + h) | B(r) \\ \text{with } r \leq s + h) \\ \leq \sup_{\zeta \in \mathbb{R}} \mathbb{P}(B(t - s - h) + \zeta \in (f(t) - h, f(t) + h)). \end{aligned}$$

Since $B(t - s - h)$ is normally distributed we know by the unimodality property of the normal distribution that,

$$\sup_{\zeta \in \mathbb{R}} \mathbb{P}(B(t - s - h) + \zeta \in (f(t) - h, f(t) + h)) \leq \mathbb{P}(B(t - s - h) \in (-h, h)).$$

Hence, we get for T_1 that,

$$\begin{aligned} T_1 &\leq \int_h^1 \int_0^{(t-h)^+} \mathbb{P}(B(t - s - h) \in (-h, h)) \mathbb{P}(B(s) \in (f(s) - h, f(s) + h), \\ &\quad \mathcal{K}_a(s, h)) d\mu(s) d\mu(t) \\ &\leq \int_0^1 \mathbf{1}_{S^\circ(h)}(r) d\mu(r) \cdot \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{2\log(1/h)}}^\infty \exp(-\frac{u^2}{2}) du \\ &\quad \cdot \int_h^1 \int_0^{(t-h)^+} \mathbb{P}(B(t - s - h) \in (-h, h)) \mathbb{P}(B(s) \in (f(s) - h, f(s) + h)) \\ &\quad d\mu(s) d\mu(t). \end{aligned}$$

Now, by applying (4),

$$T_1 \leq c_2 h^2 \int_{a\sqrt{2\log(1/h)}}^\infty \exp(-\frac{u^2}{2}) du \int_h^1 \int_0^{(t-h)^+} \frac{1}{\sqrt{s(t-s-h)}} d\mu(s) d\mu(t),$$

with some positive constant c_2 (depending on $\max_{x \in [0,1]} |f(x)|$). Further, we get

$$\begin{aligned}
T_1 &\leq c_2 h^2 \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp\left(-\frac{u^2}{2}\right) du \int_0^{1-h} \frac{1}{\sqrt{s}} \int_{s+h}^1 \frac{1}{\sqrt{(t-s-h)}} d\mu(t) d\mu(s) \\
&\leq c_2 h^2 \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp\left(-\frac{u^2}{2}\right) du \cdot S_1^2(\mu) \\
&\leq \frac{4c_2 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^2 \int_{a\sqrt{2\log(1/h)}}^{\infty} \exp\left(-\frac{u^2}{2}\right) du,
\end{aligned} \tag{7}$$

where the last step follows from Lemma 4.3, with $\eta > 1/2$.

The next step is to estimate T_2 . Again, we use the unimodality argument as before. For all $t \geq s$ and $h > 0$,

$$\begin{aligned}
\mathbb{P}(B(t) \in (f(t)-h, f(t)+h), B(s) \in (f(s)-h, f(s)+h)) \\
\leq \mathbb{P}(B(t-s) \in (-h, h)) \mathbb{P}(B(s) \in (f(s)-h, f(s)+h)) \\
\leq \mathbb{P}(B(t-s) \in (-h, h)) \mathbb{P}(B(s) \in (-h, h)).
\end{aligned}$$

Now we can use the same calculations as in [KS], p.413 to bound T_2 from above. For the sake of completeness we perform them in the following. With (4) we get that,

$$\begin{aligned}
T_2 &\leq C_2^2 h^2 \int_0^1 \int_{(t-h)^+}^t \frac{1}{\sqrt{s(t-s)}} d\mu(s) d\mu(t) \\
&= C_2^2 h^2 \left[\int_0^h \int_0^t \frac{1}{\sqrt{s(t-s)}} d\mu(s) d\mu(t) + \int_h^1 \int_{t-h}^t \frac{1}{\sqrt{s(t-s)}} d\mu(s) d\mu(t) \right] \\
&\leq C_2^2 h^2 \left[\int_0^h \frac{1}{\sqrt{s}} \int_s^h \frac{1}{\sqrt{t-s}} d\mu(t) d\mu(s) \right. \\
&\quad \left. + \int_0^1 \frac{1}{\sqrt{s}} \int_s^{(s+h) \wedge 1} \frac{1}{\sqrt{t-s}} d\mu(t) d\mu(s) \right] \\
&\leq C_2^2 h^2 \left[S_h^2(\mu) + S_1(\mu) \tilde{S}_h(\mu) \right] \\
&\leq C_2^2 h^2 \left[\frac{4 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^{2\eta-1} + \frac{4 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^{\eta-1/2} \right] \\
&\leq \frac{8C_2^2 \exp(2\eta)}{(2\eta-1)^2} A_\eta^2(\mu) h^{\eta+3/2}
\end{aligned} \tag{8}$$

Therefore, with (6), (7) and (8) we can now bound $\mathbb{E}(J_\mu^2(h, a))$ from above. There is a constant $c_3 > 0$ such that

$$\mathbb{E}(J_\mu^2(h, a)) \leq \frac{c_3 \exp(2\eta)}{(2\eta - 1)^2} A_\eta^2(\mu) (h^{\eta+3/2} \Phi + h^2 \Phi^2), \quad (9)$$

where $\Phi = \frac{1}{\sqrt{2\pi}} \int_a^\infty \sqrt{2 \log(1/h)} \exp(-\frac{u^2}{2}) du$.

The next step is applying the second moment method. First, we define the four sets,

$$\mathfrak{G}(a, h) := \left\{ t \in [0, 1] \mid \sup_{0 \leq s \leq h} \frac{|X(t+s) - X(t)|}{\sqrt{2s \log(1/s)}} > a \right\},$$

and

$$\mathfrak{G}^+(a, h) := \left\{ t \in [0, 1] \mid \sup_{\substack{0 \leq s \leq h \\ s: t \in \mathcal{S}^-(s)}} \frac{B(t+s) - B(t)}{\sqrt{2s \log(1/s)}} > a \right\},$$

and

$$\mathfrak{G}^-(a, h) := \left\{ t \in [0, 1] \mid \sup_{\substack{0 \leq s \leq h \\ s: t \in \mathcal{S}^+(s)}} \frac{B(t+s) - B(t)}{\sqrt{2s \log(1/s)}} < -a \right\},$$

and $\mathcal{Z}_h(X) := \{t \in [0, 1] : |X(t)| < h\}$. Note that $\mathfrak{G}^+(a, h) \cup \mathfrak{G}^-(a, h) \subset \mathfrak{G}(a, h)$. By Lemma 4.2, if $\eta < \dim(E)$, then there is an atomless probability measure μ on E with $A_\eta(\mu) < \infty$. Fix such a measure μ and an η such that $a^2 + 1/2 < \eta < \dim(E)$.

By the Paley-Zygmund inequality, $\mathbb{P}(J_\mu(h, a) > 0) \geq \frac{(\mathbb{E}[J_\mu(h, a)])^2}{\mathbb{E}(J_\mu^2(h, a))}$. Using the fact that $\int_x^\infty \exp(-\frac{u^2}{2}) du \geq \frac{x}{x^2+1} \exp(-\frac{x^2}{2})$ (see for instance [MP], Lemma 12.9), we see that $\Phi \geq \frac{h^{a^2} a \sqrt{\log(1/h)}}{\sqrt{\pi}(2a^2 \log(1/h)+1)}$. Now, for small enough h and some positive constant c_4 we get

$$\begin{aligned} \frac{(\mathbb{E}[J_\mu(h, a)])^2}{\mathbb{E}(J_\mu^2(h, a))} &\geq \frac{c_4(2\eta - 1)^2}{\exp(2\eta) A_\eta^2(\mu)} \left[\frac{h^{\eta-1/2}}{\Phi} + 1 \right]^{-1} \\ &\geq \frac{c_4(2\eta - 1)^2}{\exp(2\eta) A_\eta^2(\mu)} \left[h^{\eta-a^2-1/2} \frac{\sqrt{\pi}(2a^2 \log(1/h) + 1)}{a \sqrt{\log(1/h)}} + 1 \right]^{-1}. \end{aligned}$$

Since $h^{\eta-a^2-1/2} \frac{\sqrt{\pi}(2a^2 \log(1/h)+1)}{a \sqrt{\log(1/h)}}$ goes to 0 as h goes to 0, it follows that there is a number ρ depending on η for small enough h with $\liminf_{h \rightarrow 0^+} \mathbb{P}(J_\mu(h, a) > 0) > \rho > 0$. The event $J_\mu(h, a) > 0$ implies $\mathfrak{G}(a, h) \cap \mathcal{Z}_h(X) \cap E \neq \emptyset$. Note that if $h \leq h'$, then $\{\mathfrak{G}(a, h) \cap \mathcal{Z}_h(X)\} \subset \{\mathfrak{G}(a, h') \cap \mathcal{Z}_{h'}(X)\}$. $\bigcap_{h>0} \{\mathfrak{G}(a, h) \cap \mathcal{Z}_h(X)\} \cap E$ equals to

$$\left\{ t \in \mathcal{Z}(X) \cap E \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a \right\}.$$

Observe that for every $h > 0$, $\mathfrak{G}(a, h) \cap \mathcal{Z}_h(X)$ is an open subset of $[0, 1]$. We can apply Lemma 4.4 since $\overline{\text{codim}}(\mathfrak{G}(a, h) \cap \mathcal{Z}_h(X)) \leq \dim(E)$ for all small enough $h > 0$). It follows that

$$\mathbb{P}\left(\left\{t \in \mathcal{Z}(X) \cap E \mid \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{\sqrt{2h \log(1/h)}} \geq a\right\} \neq \emptyset\right) > 0.$$

□

5. PROOF OF THEOREM 1.2

We will need some notation first. An interval I is called *dyadic* if it is of the form $I = [k2^m, (k+1)2^m]$ for some integers $k \geq 0$ and m . For each positive integer m let F_m be the collection of dyadic intervals $[k2^{-m}, (k+1)2^{-m}]$ for $k = 0, \dots, 2^m - 1$ and F be the union over all such collections. For each interval $I \in F$ let $L(I)$ be a random variable that takes only the values 0 and 1. Define the sets

$$\mathcal{J}_m := \bigcup_{\substack{I \in F_m: \\ L(I)=1}} I,$$

and

$$\mathcal{J} := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \mathcal{J}_m.$$

\mathcal{J} is called *limsup fractal* since $\mathbf{1}_{\mathcal{J}} = \limsup_{m \rightarrow \infty} \mathbf{1}_{\mathcal{J}_m}$. In order to prove Theorem 1.2 we will show a lower bound on the Hausdorff dimension of a certain limsup fractal. For more on this method see for instance [MP].

Fix an $\epsilon > 0$ and an integer $m > 0$. For an interval $I = [t_I, s_I]$ of the form $[k2^{-m}, (k+1)2^{-m}]$ we set $L(I) = 1$ if $|B(t_I + m2^{-m}) - B(t_I)| > a(1 + \epsilon)\sqrt{m2^{-m+1} \log(m^{-1}2^m)}$ holds.

Now we want to show that the set \mathcal{J} associated with this family of random variables $\{L(I), I \in F\}$ is contained in a set of points fulfilling that at least “half of the points” are a -fast times. Then, a lower bound of the Hausdorff dimension of the set of \mathcal{J} is also a lower bound of the set of a -fast times of the process X .

Note that there is a constant $c_1 > 0$ such that for all $s, t \in [0, 2]$ with $|s - t| \leq h'$ with random $h' > 0$,

$$|B(s) - B(t)| \leq c_1 \sqrt{|s - t| \log \frac{1}{|s - t|}},$$

almost surely (see Theorem 1.12 of [MP]). Let $t \in \mathcal{J}$ and also $t \in I = [t_I, s_I] \in F_m$ with $L(I) = 1$. Then, by the triangle inequality it follows

$$\begin{aligned} & |B(t + m2^{-m}) - B(t)| \\ & \geq |B(t_I + m2^{-m}) - B(t_I)| - |B(t + m2^{-m}) - B(t_I + m2^{-m})| - |B(t_I) - B(t)|. \end{aligned}$$

Now we see that for m (larger than some random $m' > 0$ and) large enough such that $a\epsilon\sqrt{2m\log(m^{-1}2^m)} \geq 2c_1\sqrt{\log 2^m}$ the following inequalities hold,

$$\begin{aligned} |B(t_I + m2^{-m}) - B(t_I)| - |B(t + m2^{-m}) - B(t_I + m2^{-m})| - |B(t_I) - B(t)| \\ \geq a(1 + \epsilon)\sqrt{m2^{-m+1}\log(m^{-1}2^m)} - 2c_1\sqrt{2^{-m}\log 2^m} \\ \geq a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}. \end{aligned}$$

This event happens for infinitely many m 's. Therefore, t is an a -fast time of B .

Further, we define a process \hat{B} depending on the Brownian motion B by tossing a coin,

$$\hat{B} = \begin{cases} B, & \text{with probability } 1/2, \\ -B, & \text{with probability } 1/2. \end{cases}$$

If the time t is an a -fast time of B , then it is also an a -fast time of \hat{B} . Note that if

$$|\hat{B}(t + m2^{-m}) - \hat{B}(t)| \geq a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$$

holds, then conditional on this the event

$$\hat{B}(t + m2^{-m}) - \hat{B}(t) \geq a\sqrt{m2^{-m+1}\log(m^{-1}2^m)},$$

happens with probability of at least $1/2$ and

$$\hat{B}(t + m2^{-m}) - \hat{B}(t) \leq -a\sqrt{m2^{-m+1}\log(m^{-1}2^m)},$$

happens with probability of at least $1/2$ as well. Therefore, we see that the event that $\hat{B}(t + m2^{-m}) - \hat{B}(t) \geq a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$ if $f(t + m2^{-m}) - f(t) \leq 0$ or $\hat{B}(t + m2^{-m}) - \hat{B}(t) \leq -a\sqrt{m2^{-m+1}\log(m^{-1}2^m)}$ if $f(t + m2^{-m}) - f(t) > 0$ happens with probability of at least $1/2$. Since \hat{B} is also a Brownian motion, it follows that t is an a -fast time of the process X with probability of at least $1/2$.

Now set $\dim \mathcal{J} = \alpha$ (we actually know by Orey, Taylor (1) and Theorem 5.1 that $\dim \mathcal{J} = 1 - a^2$ almost surely). Let $\epsilon > 0$, then there exists a probability measure μ on \mathcal{J} such that the energy

$$\mathbb{E}\left(\int_{[0,1]} \int_{[0,1]} \frac{1}{|x - y|^{\alpha - \epsilon}} d\mu(x) d\mu(y)\right) < \infty,$$

see for instance [MP], p.113 or [Matt]. Define

$$\hat{\mathcal{J}} := \{t \in \mathcal{J} | t \text{ is an } a\text{-fast time of } X\},$$

and a probability measure on $\hat{\mathcal{J}}$ by $\mu'(A) = \frac{\mu(A)}{\mu(\hat{\mathcal{J}})}$, where A are measurable sets with respect to μ . Note that

$$\mu(\hat{\mathcal{J}}) = \int_{[0,1]} \mathbb{P}(t \text{ is an } a\text{-fast time of } X \mid \mathfrak{F}) d\mu(t),$$

where \mathfrak{F} is the sigma algebra of B . Then $\mu(\hat{\mathcal{J}}) \geq \frac{1}{2}\mu(\mathcal{J}) = \frac{1}{2}$. Therefore,

$$\begin{aligned} \mathbb{E}\left(\int_{[0,1]} \int_{[0,1]} \frac{1}{|x-y|^{\alpha-\epsilon}} d\mu'(x) d\mu'(y)\right) \\ \leq \mathbb{E}\left(4 \cdot \int_{[0,1]} \int_{[0,1]} \frac{1}{|x-y|^{\alpha-\epsilon}} d\mu(x) d\mu(y)\right) < \infty. \end{aligned}$$

This implies $\dim \hat{\mathcal{J}} > \alpha - \epsilon$ almost surely (see Theorem 4.27 of [MP]), and by letting $\epsilon \downarrow 0$, it follows that $\dim \hat{\mathcal{J}} \geq \dim \mathcal{J}$ almost surely.

The rest of the proof is the same as in [MP] and we give the details for the sake of completeness. The next step is to bound the first moment of $L(I)$ for $I \in F_m$ from below. Note that

$$\begin{aligned} \mathbb{P}(L(I) = 1) &\geq \mathbb{P}(B(t_I + m2^{-m}) - B(t_I) > a(1+\epsilon)\sqrt{m2^{-m+1}\log(m^{-1}2^m)}) \\ &\geq \mathbb{P}(B(1) > a(1+\epsilon)\sqrt{2\log(m^{-1}2^m)}) \\ &\geq 2^{-ma^2(1+\epsilon)^3}, \end{aligned}$$

for large enough m and where the last step follows from the fact that

$$\int_x^\infty \exp\left(-\frac{u^2}{2}\right) du \geq \frac{x}{x^2+1} \exp\left(-\frac{x^2}{2}\right),$$

(see for instance [MP], Lemma 12.9) and $\frac{a(1+\epsilon)\sqrt{2\log(m^{-1}2^m)}}{\sqrt{2\pi(1+2a^2(1+\epsilon)^2\log(m^{-1}2^m))}} \exp(-a^2(1+\epsilon)^2\log(m^{-1}2^m)) \geq 2^{-ma^2(1+\epsilon)^3}$ for sufficiently large enough m .

In order to prove Theorem 1.2 we will apply the following theorem.

Theorem 5.1 (Theorem 10.6 of [MP]). *Let \mathcal{J} be a limsup fractal associated to the family of random variables $\{L(I), I \in F\}$. Suppose $p_k := \mathbb{P}(L(I) = 1)$ is the same for all $I \in F_k$, and for an interval $I \in F_m$ let $M_n(I) := \sum_{I' \subset I, I' \in F_n} L(I')$ with $m \leq n$. If there are $\eta_n \geq 1$ and $\gamma \in (0, 1)$ such that*

$$\mathbb{V}(M_n(I)) \leq \eta_n \mathbb{E}(M_n(I)) = \eta_n p_n 2^{n-m},$$

and also

$$\lim_{n \rightarrow \infty} 2^{(\gamma-1)n} \cdot \frac{\eta_n}{p_n} = 0,$$

then $\dim \mathcal{J} \geq \gamma$, almost surely.

In order to be able to apply Theorem 5.1 it is left to bound the variance $\mathbb{V}(M_n(I))$ from above. To achieve this, we see that

$$\begin{aligned} \mathbb{E}(M_n^2(I)) &= \sum_{\substack{I_1, I_2 \subset I, \\ I_1, I_2 \in F_n}} \mathbb{E}[L(I_1)L(I_2)] \\ &\leq \sum_{\substack{I_1 \subset I, \\ I_1 \in F_n}} \left[(2n+1)\mathbb{E}(L(I_1)) + \mathbb{E}(L(I_1)) \sum_{\substack{I_2 \subset I, \\ I_2 \in F_n}} \mathbb{E}(L(I_2)) \right], \end{aligned}$$

where we used that the random variables $L(I_1)$ and $L(I_2)$ are independent if distance of the two intervals I_1 and I_2 is at least $n2^{-n}$, and further the trivial estimate $\mathbb{E}[L(I_1)L(I_2)] \leq \mathbb{E}[L(I_1)]$. It follows that

$$\mathbb{V}(M_n(I)) = \mathbb{E}(M_n^2(I)) - \mathbb{E}(M_n(I))^2 \leq \sum_{\substack{I_1 \subset I, \\ I_1 \in F_n}} (2n+1)p_n = 2^{n-m}(2n+1)p_n.$$

Applying Theorem 5.1 for $\gamma < 1 - a^2(1 + \epsilon)^3$, the claim follows by letting $\epsilon \downarrow 0$. ■

ACKNOWLEDGMENTS

The author thanks gratefully Michael Scheutzow for very fruitful discussions and advice.

REFERENCES

- [ABPR] T. Antunović, K. Burdzy, Y. Peres, and J. Ruscher. *Isolated zeros for Brownian motion with variable drift*. Electronic Journal of Probability. Vol. 16, No. 65: 1793–1814, 2011.
- [Haw] John Hawkes. Trees generated by a simple branching process. *J. London Math. Soc.* (2), 24(2):373–384, 1981.
- [Kah] Jean-Pierre Kahane. *Some random series of functions*, volume 5 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1985.
- [K74] R. Kaufman. *Large increments of Brownian Motion*. Nagoya Math. J. 56, 139–145, 1974.
- [Kho02] D. Khoshnevisan. *Multiparameter processes: an introduction to random fields*. Springer monographs in mathematics, Springer Verlag, 2002.
- [Kho03] D. Khoshnevisan. *The Codimension of the Zeros of a Stable Process in Random Scenery*. Séminaire de Probabilités XXXVII, Lecture Notes in Mathematics - Springer Verlag Issue 1832, 236–245, 2003.
- [KPX] D. Khoshnevisan, Y. Peres, Y. Xiao. *Limsup Random Fractals*. Electronic Journal of Probability. Vol. 5, No. 4: 1–24, 2000.
- [KS] D. Khoshnevisan, Z. Shi. *Fast Sets and Points for Fractional Brownian Motion*. Séminaire de Probabilités XXXIV, Lecture Notes in Mathematics - Springer Verlag Issue 1729, 393–416, 2000.
- [Loud] W. S. Loud. Functions with prescribed Lipschitz condition. *Proc. Amer. Math. Soc.*, 2:358–360, 1951.
- [Matt] P. Mattila. *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*. Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.
- [MP] P. Mörters and Y. Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.
- [MP53] Imanuel Marx and George Piranian. Lipschitz functions of continuous functions. *Pacific J. Math.*, 3:447–459, 1953.
- [OT74] S. Orey and S.J. Taylor. *How often on a Brownian path does the law of the iterated logarithm fail?* Proc. London Math. Soc. 28, 174–192 (1974).
- [R12] J. Ruscher. *Cantor set zeros of one-dimensional Brownian motion minus Cantor function*. Preprint, 2012.

- [RY] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [Tay86] S. J. Taylor. *The measure theory of random fractals*. Math. Proc. Camb. Phil. Soc. 100, 383–406 (1986).

FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI
136, D-10623 BERLIN, AND
INSTITUTE FOR ADVANCED STUDY, EINSTEIN DRIVE, PRINCETON, NJ 08540
E-mail address: `ruscher@math.tu-berlin.de`